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On the stability against shear waves of steady flows of non-linear viscoelastic fluids

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A shear-acceleration wave is a propagating singular surface across which the velocity vector and the normal component of the acceleration are continuous, while the tangential component $\dot{v}$ of the acceleration suffers a jump discontinuity $[\dot{v}]$. We here discuss plane-rectilinear shearing flows of general, non-linear, incompressible simple fluids with fading memory. Working within the framework of such planar motions, we derive a general and exact formula for the time-dependence of the amplitude $a = [\dot{v}]$ of a shear-acceleration wave propagating into a region undergoing a steady but not necessarily homogeneous shearing flow. When this expression is specialized to the case in which the velocity gradient is constant in space ahead of the wave, it assumes a form familiar in the theory of longitudinal acceleration waves in compressible materials with fading memory (cf., e.g., Coleman & Gurtin 1965, equation (4.12)).

In earlier work (1965) we observed that a planar shear-acceleration wave cannot grow in amplitude if it is propagating into a fluid in a state of equilibrium. It is clear from our present results that if the fluid ahead of the wave is being sheared, $|a(t)|$ not only increases, but can approach infinity in a finite time, provided $a(0)$ is of proper sign and $|a(0)|$ exceeds a certain critical amplitude. We expect this critical amplitude to decrease as the rate of shear ahead of the wave is increased.

1. Introduction

We here attempt to develop an exact theory of the growth and decay of surfaces of discontinuity in non-linear, incompressible, viscoelastic fluids. After using Coleman & Noll's (1961 *a*) analysis of rectilinear shearing motions of general simple fluids to obtain reduced forms of the dynamical equations, we apply recently found methods of calculating the amplitude of one-dimensional acceleration waves in materials with memory.† We find, as was mentioned in the summary, that although a planar shear-acceleration wave propagating into a region in equilibrium is damped out, when such a wave advances into a region undergoing a steady shearing flow, the wave amplitude can, if the conditions are right,

† See, for example, Coleman & Gurtin (1965), Dunwoody & Dunwoody (1965), Varley (1965). It is possible to obtain our present results by specializing and combining theorems given by Coleman, Greenberg & Gurtin (1966), but we have found that for fluids it is easier and more instructive to start again from first principles.

approach infinity in a finite time. This result suggests an apparently new type of instability for steady shearing flows of viscoelastic fluids: the breakdown of a steady flow by the rapid growth of a disturbance involving a jump in the acceleration. In terms more suggestive than precise, the greater the rate of shear in a steady rectilinear flow, the less resistant is the steady flow to shear waves.

In §5 we derive and briefly discuss the linear field equations describing first-order perturbations about a simple shearing flow. On contrasting the exact theory of §§6 and 7 with the linearized theory of §5, it becomes apparent that only the exact theory can yield the instabilities we have found. The reciprocal of the critical amplitude for the growth of an acceleration wave is proportional to the 'second-order instantaneous modulus', a material parameter which does not occur in linearized theories.

When a molten polymer is extruded through a narrow tube or channel, the desired steady laminar flow is observed to break down at a critical rate of discharge (Nason 1945; Spencer & Dillon 1949; Tordella 1956, 1957; Bagley 1957, 1963). Although this failure phenomenon, called 'melt fracture' or 'elastic turbulence', is often associated with entrance effects, there is experimental evidence that an already established steady shearing flow can break down *within* a narrow tube (Benbow, Charley & Lamb 1961; Tordella 1963). It appears to us possible that the onset of melt fracture may be due to the rapid growth of initially feeble shear-acceleration waves of proper sign propagating into regions undergoing high rates of shear. It is likely that the attainment of infinite amplitude by a shear-acceleration wave signifies the formation of a vortex sheet across which there is a non-zero jump in the tangential component of the velocity. When using conventional extruders and dies, it is probably difficult, if not impossible, to distinguish, on the one hand, the formation of vortex sheets near the boundaries in accord with (6.35) from, on the other hand, a failure of adherence to the boundaries governed only by an *ad hoc* slip condition. We hope, however, that an experimenter will find a way to make the distinction and test our theory. The information summarized in table 1 on page 180 may be useful for this purpose.

2. Concepts from the general theory of simple fluids

We consider a flowing fluid body and let $\boldsymbol{\xi}$ be the place in space occupied at time σ by that material point X which occupies the place $\mathbf{x}$ at time t. For the dependence of $\boldsymbol{\xi}$ on $\mathbf{x}$, t and σ we write

$$\boldsymbol{\xi} = \boldsymbol{\chi}_t(\mathbf{x}, \sigma). \tag{2.1}$$

The gradient, with respect to $\mathbf{x}$, of the deformation function $\boldsymbol{\chi}_t$,

$$\mathbf{F}(\sigma) = \text{grad}_{\mathbf{x}}\boldsymbol{\chi}_t(\mathbf{x}, \sigma), \tag{2.2}$$

is the *relative deformation gradient* for X at time σ. Of importance in the theory of viscoelastic fluids is the function $\mathbf{C}^t$, defined by

$$\mathbf{C}^t(s) = \mathbf{F}(t-s)^T\mathbf{F}(t-s) \quad (0 \leqslant s < \infty). \tag{2.3}$$

This function, called the *relative strain-history* of X up to time t, maps $[0, \infty)$ into the space of symmetric tensors. For each $s \geqslant 0$, $\mathbf{C}^t(s)$ is the right Cauchy–Green tensor at X at time $t-s$, computed relative to the configuration at time t. Clearly

$$\mathbf{C}^t(0) = \mathbf{1}, \tag{2.4}$$

with $\mathbf{1}$ the unit tensor.

The stress $\mathbf{T}(t)$ at a material point X of an incompressible *simple fluid* is determined, to within a hydrostatic pressure, by the relative strain-history of X. Thus, each incompressible simple fluid is characterized by a constitutive functional $\mathscr{F}$ such that (cf. Noll 1958; Coleman & Noll 1959, 1961)

$$\mathbf{T}(t) = -p\mathbf{1} + \underset{s=0}{\overset{\infty}{\mathscr{F}}} (\mathbf{C}^t(s)), \tag{2.5}$$

where p is the indeterminate hydrostatic pressure. It follows from the principle of material objectivity (Noll 1958) that the functional $\mathscr{F}$ must obey the following identity in $\mathbf{C}^t$ for each constant orthogonal tensor $\mathbf{Q}$:

$$\underset{s=0}{\overset{\infty}{\mathscr{F}}} (\mathbf{Q}\mathbf{C}^t(s)\,\mathbf{Q}^T) = \mathbf{Q}\underset{s=0}{\overset{\infty}{\mathscr{F}}} (\mathbf{C}^t(s))\,\mathbf{Q}^T. \tag{2.6}$$

We do not assume, nor do we expect $\mathscr{F}$ to be a linear functional.

Flow problems are usually stated in terms of the velocity field $\mathbf{v}$. It is easy to show that when the field $\mathbf{v} = \mathbf{v}(\mathbf{x}, \sigma)$ is specified for all times $\sigma \leqslant t$, the function $\mathbf{C}^t$ is determined at each point. Indeed, if, for each $\mathbf{x}, \boldsymbol{\xi}(\,.\,)$ is that solution of the differential equation

$$\frac{d}{d\sigma}\boldsymbol{\xi}(\sigma) = \mathbf{v}(\boldsymbol{\xi}(\sigma), \sigma), \tag{2.7}$$

which satisfies the end condition

$$\boldsymbol{\xi}(t) = \mathbf{x}, \tag{2.8}$$

then

$$\boldsymbol{\xi}(\sigma) = \boldsymbol{\chi}_t(\mathbf{x}, \sigma) \quad (\sigma \leqslant t). \tag{2.9}$$

Thus, $\mathbf{v}$ determines the deformation function $\boldsymbol{\chi}_t$ in (2.1), and hence, in view of (2.2) and (2.3), $\mathbf{v}$ determines $\mathbf{C}^t$. (Cf. Coleman & Noll 1961*a*, § 1.)

Since only isochoric motions are possible in incompressible fluids, the velocity field here obeys the condition

$$\operatorname{div} \mathbf{v} \equiv 0, \tag{2.10}$$

which is equivalent to

$$|\det \mathbf{C}^t(s)| \equiv 1. \tag{2.11}$$

3. Basic properties of rectilinear shearing flows

If in a fixed Cartesian co-ordinate system x, y, z, the velocity field has the form

$$v^x = 0, \quad v^y = v(x,t), \quad v^z = 0, \tag{3.1}$$

then we say that the motion is a *rectilinear shearing flow*. For such a flow, (2.10) is automatically satisfied, and the differential equation (2.7) with the end condition (2.8) has the solution

$$\xi^x(\sigma) = x, \quad \xi^y(\sigma) = y + \int_t^\sigma v(x,r)\,dr, \quad \xi^z(\sigma) = z. \tag{3.2}$$

Here $\xi^x(\sigma)$, $\xi^y(\sigma)$, $\xi^z(\sigma)$ are the Cartesian co-ordinates at time σ of the material point X which has the co-ordinates x, y, z at time t. Employing (2.1)–(2.3), one may easily verify that (3.2) yields the following expression for the matrix of the components of $\mathbf{C}^t(s)$:

$$[\mathbf{C}^t(s)] = \begin{bmatrix} 1+\lambda^t(s)^2 & \lambda^t(s) & 0 \\ \lambda^t(s) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.3}$$

with λ^t the real-valued function on $[0, \infty)$ defined by

$$\lambda^t(s) = -\int_{t-s}^{t} \partial_x v(x,\sigma)\, d\sigma \quad (0 \leqslant s < \infty), \tag{3.4}$$

and called the *relative shearing history* at x. Here ∂_x denotes the partial derivative with respect to x.

Coleman & Noll (1961 *a*, § 5) have shown that (2.5), (2.6) and (3.3) imply that in a rectilinear shearing flow the components of the stress obey the relations

$$\left.\begin{aligned} T^{xy}(t) &= \underset{s=0}{\overset{\infty}{\mathfrak{t}}}\,(\lambda^t(s)), \quad T^{xx}(t) - T^{zz}(t) = \underset{s=0}{\overset{\infty}{\mathfrak{s}_1}}\,(\lambda^t(s)), \\ T^{yy}(t) - T^{zz}(t) &= \underset{s=0}{\overset{\infty}{\mathfrak{s}_2}}\,(\lambda^t(s)), \quad T^{xz} = T^{yz} = 0, \end{aligned}\right\} \tag{3.5}$$

where $\mathfrak{t}$, $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are real-valued functionals obeying the identities

$$\left.\begin{aligned} \underset{s=0}{\overset{\infty}{\mathfrak{t}}}\,(-\lambda^t(s)) &= -\underset{s=0}{\overset{\infty}{\mathfrak{t}}}\,(\lambda^t(s)), \\ \underset{s=0}{\overset{\infty}{\mathfrak{s}_i}}\,(-\lambda^t(s)) &= \underset{s=0}{\overset{\infty}{\mathfrak{s}_i}}\,(\lambda^t(s)), \quad (i = 1, 2). \end{aligned}\right\} \tag{3.6}$$

The functionals $\mathfrak{t}$, $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are determined when $\mathscr{F}$ in (2.5) is specified and are independent of the direction of shearing.

We assume, as is usual, that the long-range body forces $\mathbf{b}$ acting on the fluid have a single-valued potential ψ:

$$\mathbf{b} = -\operatorname{grad} \psi. \tag{3.7}$$

Therefore, it follows from (3.1) and (3.4)–(3.6) that the dynamical equations,

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \tag{3.8}$$

with ρ the mass density, here take the simple form†

$$\partial_x T^{xy} - \partial_y(p + \rho\psi) = \rho \partial_t v, \quad \partial_x T^{xx} - \rho \partial_x \psi = 0, \quad \partial_z(p+\rho\psi) = 0. \tag{3.9}$$

An elementary analysis shows that these equations are equivalent to asserting) that

$$\partial_x T^{xy} + \alpha(t) = \rho \partial_t v \tag{3.10}$$

and

$$T^{xx} = \alpha(t) y + \beta(t) + \rho\psi, \tag{3.11}$$

where α and β are functions of t only. The total driving force exerted on a column

† See Coleman & Noll (1959) for the special case of steady shearing flow and Truesdell & Noll (1965, § 11) for the present general case.

of fluid lying between the planes $y = y_1$ and $y = y_2$ is (cf. Coleman & Noll 1959, §6)

$$f(t) = \left(\int_{\mathscr{A}} T^{yy}\, dA\right)_{y=y_2} - \left(\int_{\mathscr{A}} T^{yy}\, dA\right)_{y=y_1} + \int_{y_1}^{y_2}\int_{\mathscr{A}} \rho b^y\, dA\, dy$$

$$= \int_{\mathscr{A}} \{(T^{yy} - \rho\psi)_{y=y_2} - (T^{yy} - \rho\psi)_{y=y_1}\}\, dA. \tag{3.12}$$

By (3.4), $(3.5)_2$ and (3.11),

$$f(t) = \int_{\mathscr{A}} (y_2 - y_1)\alpha(t)\, dA = (y_2 - y_1) A\alpha(t), \tag{3.13}$$

where A is the cross-sectional area of the column. Since $A(y_2 - y_1)$ is the volume of the column of fluid considered, it follows from (3.13) that $\alpha(t)$ in (3.10) and (3.11) is the driving force per unit volume.

In view of (3.4) and $(3.5)_1$, it is clear that when the specific driving force $\alpha = \alpha(t)$ is specified, (3.10) becomes a functional-differential equation for the function v in (3.1). We may write this equation in the form

$$\partial_x \underset{s=0}{\overset{\infty}{\mathfrak{t}}} (\lambda^t(s)) + \alpha = \rho \partial_t v(x, t). \tag{3.14}$$

For the rest of this section, we assume that α is constant in time.

As we intend to study the dynamical stability of steady-flow solutions of (3.14), let us now briefly review the salient properties of such solutions.

For a steady rectilinear shearing flow, $\partial_t v \equiv 0$, and (3.4) reduces to

$$\lambda^t(s) = -\kappa s, \tag{3.15}$$

where

$$\kappa = \frac{d}{dx} v(x) \tag{3.16}$$

is called the *rate of shear* at x. Thus, if the flow is known to be steady, (3.14) yields

$$\frac{d}{dx}\tau(\kappa) + \alpha = 0, \tag{3.17}$$

where τ is the *shear-stress function* familiar in the theory of steady viscometric flows† and defined by

$$\tau(\kappa) = \underset{s=0}{\overset{\infty}{\mathfrak{t}}} (-s\kappa). \tag{3.18}$$

It follows from $(3.6)_1$ that this function τ, mapping the real numbers into the real numbers, is an *odd* function:

$$\tau(-\kappa) = -\tau(\kappa), \quad \tau(0) = 0. \tag{3.19}$$

We assume that τ is invertible throughout its domain and denote its inverse function by τ^{-1}. Clearly, (3.17) implies that

$$\tau(\kappa) = -\alpha x + \beta$$

† See, for example, Coleman & Noll (1959). For a survey see Coleman, Markovitz & Noll (1966).

with β constant. Hence, in each steady rectilinear shearing flow the rate of shear κ is given by an equation of the form

$$\kappa = \tau^{-1}(-\alpha x+\beta). \tag{3.20}$$

If the driving force α is zero, (3.20) yields κ independent of x, and the velocity function v takes the form

$$v = V+\kappa x, \tag{3.21}$$

with κ and V constants. A steady rectilinear shearing flow for which v in (3.1) has the simple form (3.21) is called a *simple shearing flow*. A simple shearing flow with $\kappa = 0$ may be called a *state of equilibrium*.

Steady channel flow is a steady rectilinear shearing flow between two infinite parallel plates which are both at rest and parallel to the (y,z)-plane. If we let d be the distance between the plates and place the (y,z)-plane halfway between the plates, then the adherence condition yields the boundary conditions

$$v(+\tfrac{1}{2}d) = v(-\tfrac{1}{2}d) = 0. \tag{3.22}$$

Since τ^{-1} is an odd function, (3.22) is compatible with (3.20) only if $\beta = 0$, and we have

$$\frac{d}{dx}v(x) = \kappa(x) = -\tau^{-1}(\alpha x). \tag{3.23}$$

Integration yields the velocity function v for steady channel flow (cf., Coleman & Noll 1959):

$$v(x) = \int_x^{\frac{1}{2}d} \tau^{-1}(\alpha x)\,dx. \tag{3.24}$$

4. Assumptions of smoothness for constitutive functionals

The postulate of fading memory introduced by Coleman & Noll (1960, 1961*b*) asserts that constitutive functionals, such as $\mathscr{F}$ in (2.5), have continuous differentials with respect to a particular norm on a space of histories $\mathbf{C}^t$. This norm is constructed in such a way that the values $\mathbf{C}^t(s)$ of $\mathbf{C}^t$ at large s (i.e. occurring in the 'distant past') receive less weight than the values at small s. We here use this postulate, but since our present study is restricted to shearing motions governed by the functional-differential equation (3.14), it is not necessary for us to discuss the theory of fading memory in full generality. We here confine our attention to the scalar-valued functional $\mathfrak{t}$.

The functions in the domain of $\mathfrak{t}$ are the relative shearing histories λ^t. It follows from $(3.4)_1$ that these shearing histories are functions on $[0,\infty)$ obeying

$$\lambda^t(0) = 0, \tag{4.1}$$

and, therefore, the value

$$\underset{s=0}{\overset{\infty}{\mathfrak{t}}}\,(\lambda^t(s))$$

of $\mathfrak{t}$ is determined by the restriction of λ^t to $(0,\infty)$. Thus, in a discussion of the functional $\mathfrak{t}$, we need not distinguish between a relative shearing history and its restriction to $(0,\infty)$.

Let h be a fixed *influence function*, i.e. a positive, monotone-decreasing, measurable function with $sh(s)$ square-integrable over $(0,\infty)$, and let $\mathscr{H}$ denote the set of measurable, real-valued functions g on $(0,\infty)$ for which $\|g\|$, defined by

$$\|g\|^2 = \int_0^\infty g(s)^2 h(s)^2 ds, \tag{4.2}$$

is finite.

In our present context, the postulate of fading memory asserts that there must exist an influence function h such that $\mathfrak{t}$ has $\mathscr{H}$ for its domain and is twice continuously differentiable in the following sense: for each g in $\mathscr{H}$ there exists on $\mathscr{H}$ a bounded linear form, $\delta\mathfrak{t}(g|\,.\,)$, and a bounded, symmetric, bilinear form, $\delta^2\mathfrak{t}(g|\,.\,,.\,)$, such that

$$\underset{s=0}{\overset{\infty}{\mathfrak{t}}}\,(g(s)+l(s)) = \underset{s=0}{\overset{\infty}{\mathfrak{t}}}\,(g(s)) + \underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}}\,(g(s)|l(s)) + \underset{s=0}{\overset{\infty}{\delta^2\mathfrak{t}}}\,(g(s)|l(s),l(s)) + o(\|l\|^2). \tag{4.3}$$

We assume that each of the functionals $\delta\mathfrak{t}(\,.\,|\,.\,)$ and $\delta^2\mathfrak{t}(\,.\,|\,.\,,.\,)$ is jointly continuous in all its arguments.

The function space $\mathscr{H}$ forms a Hilbert space with the inner-product

$$\langle f,l\rangle = \int_0^\infty f(s)\,l(s)\,h(s)^2 ds. \tag{4.4}$$

Since

$$\underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}}\,(g(s)|l(s)),$$

as a function of l in $\mathscr{H}$, is assumed to be both linear and continuous, this function can be represented as an inner-product with a function K in $\mathscr{H}$:

$$\underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}}\,(g(s)|l(s)) = \int_0^\infty K(s)\,l(s)\,h(s)^2 ds. \tag{4.5}$$

Of course, K depends on the function g; putting

$$G'(s) = K(s)\,h(s)^2, \tag{4.6}$$

we assume that the mapping $g \to G'(s)$ is, for each s, a continuous functional over $\mathscr{H}$ and that, for each g, G' has a bounded derivative G''. We further assume that the mapping $g \to G''h^{-2}$ carries $\mathscr{H}$ into itself and is continuous.

If g has the special form $g(s) \equiv -\kappa s$, the dependence of $G'(s)$ on g reduces to a dependence on κ which we may indicate by writing $G'(\kappa,s)$.† It follows from (4.5) and (4.6) that

$$\underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}}\,(-\kappa s|l(s)) = \int_0^\infty G'(\kappa,s)\,l(s)\,ds. \tag{4.7}$$

Since K belongs to $\mathscr{H}$, the function $G(\kappa,.)$, defined by

$$G(\kappa,s) = -\int_s^\infty G'(\kappa,s)\,ds = -\int_s^\infty K(s)\,h(s)^2 ds \quad (0 < s < \infty), \tag{4.8}$$

exists and

$$\lim_{s\to\infty} G(\kappa,s) = 0; \tag{4.9}$$

† Since we assume that $sh(s)$ is square-integrable, functions of the form $g(s) \equiv -\kappa s$ are automatically in $\mathscr{H}$.

$G(\kappa, .)$ is called the *stress-relaxation function* (for direction-preserving perturbations about the steady shear κ). We assume, as appears natural, the inequalities

$$E(\kappa) \overset{\text{def}}{=} \lim_{s\to 0} G(\kappa, s) > 0, \tag{4.10}$$

$$G'(\kappa, 0) \overset{\text{def}}{=} \lim_{s\to 0} G'(\kappa, s) \leqslant 0. \tag{4.11}$$

The quantity $E(\kappa)$ defined in $(4.10)_1$ is the *instantaneous tangent modulus* (for shearing perturbations about κ). Putting for l in (4.7) the constant function on $(0, \infty)$ with value 1, we find that (4.8) yields

$$\overset{\infty}{\underset{s=0}{\delta \mathfrak{t}}} \, (-\kappa s|1) = \int_0^\infty G'(\kappa, s)\, ds = -\lim_{s\to 0} G(\kappa, s). \tag{4.12}$$

Hence we have the following alternative expression for $E(\kappa)$:

$$E(\kappa) = -\overset{\infty}{\underset{s=0}{\delta \mathfrak{t}}} \, (-\kappa s|1). \tag{4.13}$$

It follows from the identity $(3.6)_1$ that $G'(\kappa, s)$ and $G(\kappa, s)$ are, for each s, even functions of κ:

$$G'(-\kappa, s) = G'(\kappa, s), \quad G(-\kappa, s) = G(\kappa, s). \tag{4.14}$$

Therefore, we have

$$E(\kappa) = E(-\kappa) \tag{4.15}$$

and

$$\frac{d}{d\kappa} E(\kappa)\big|_{\kappa=0} = 0. \tag{4.16}$$

The number $\tilde{E}(\kappa)$, defined by

$$\tilde{E}(\kappa) = \overset{\infty}{\underset{s=0}{\delta^2 \mathfrak{t}}} \, (-\kappa s|1, 1), \tag{4.17}$$

is the *second-order instantaneous modulus* (for shearing perturbations about κ). The identity $(3.6)_1$ implies that $\tilde{E}(\kappa)$ is an odd function of κ:

$$\tilde{E}(-\kappa) = -\tilde{E}(\kappa). \tag{4.18}$$

Hence

$$\tilde{E}(0) = 0; \tag{4.19}$$

i.e. the second-order instantaneous modulus is zero for shearing perturbations about a state of equilibrium (cf. Coleman & Gurtin 1965, §6).

We have assumed that $E(\kappa)$ is strictly positive for all values of κ including zero. We may now consider two possible types of behaviour for the function $\tilde{E}(.)$. If

$$\kappa \neq 0 \Rightarrow \kappa \tilde{E}(\kappa) > 0, \tag{4.20}$$

then we say that the fluid is *shear-stiffening*. Fluids for which

$$\kappa \neq 0 \Rightarrow \kappa \tilde{E}(\kappa) < 0, \tag{4.21}$$

we call *shear-softening*. Of course, a fluid may be neither stiffening nor softening, but if $(d/d\kappa) E(\kappa)|_{\kappa=0} \neq 0$, then the fluid is either shear-stiffening or shear-soften ing in an appropriate neighbourhood of equilibrium.

5. On a linear analysis of perturbations

For a simple shearing flow with velocity function $\mathbf{v}_0$ and rate of shear κ_0, the equations (3.1) take the form

$$\left.\begin{aligned} v_0^x = 0, \quad v_0^y &= v_0(x), \quad v_0^z = 0, \\ \text{with} \qquad v_0(x) &= C + \kappa_0 x, \quad V, \kappa_0 = \text{constant}. \end{aligned}\right\} \tag{5.1}$$

As an illustration of a straightforward application of the smoothness assumptions laid down in the previous section, we here derive the linear field equations which describe first-order perturbations of the flow (5.1), supposing that the perturbed flows are rectilinear shearing flows of the type (3.1) and that the specific driving force α is held zero. Thus, we write

$$v(x,t) = v_0(x) + \epsilon v_1(x,t) \tag{5.2}$$

for the velocity function v in (5.1), and we seek, to within terms $O(\epsilon)$, the form taken by the equation (3.14) with $\alpha = 0$.

The present calculations can be simplified by first differentiating (3.14) with respect to t:

$$\partial_t \partial_x \mathop{\mathfrak{t}}_{s=0}^{\infty} (\lambda^t(s)) = \rho\, \partial_t^2 v(x,t) = \epsilon\rho\, \partial_t^2 v_1(x,t). \tag{5.3}$$

Here (3.4) becomes

$$\left.\begin{aligned} \lambda^t(s) &= -\kappa_0 s + \epsilon\omega^t(s), \\ \omega^t(s) &= -\int_{t-s}^{t} \partial_x v_1(x,\sigma)\, d\sigma, \end{aligned}\right\} \tag{5.4}$$

and, therefore,

$$\left.\begin{aligned} \partial_x \lambda^t(s) &= \epsilon\, \partial_x \omega^t(s) = -\epsilon \int_{t-s}^{t} \partial_x^2 v_1(x,\sigma)\, d\sigma, \\ \partial_t \partial_x \lambda^t(s) &= \epsilon\, \partial_t \partial_x \omega^t(s) = \epsilon[\partial_x^2 v_1(x, t-s) - \partial_x^2 v_1(x,t)]. \end{aligned}\right\} \tag{5.5}$$

Thus, (4.3) and the chain rule yield

$$\left.\begin{aligned} \partial_x \mathop{\mathfrak{t}}_{s=0}^{\infty} (\lambda^t(s)) &= \mathop{\delta\mathfrak{t}}_{s=0}^{\infty} (\lambda^t(s) | \partial_x \lambda^t(s)) = \epsilon \mathop{\delta\mathfrak{t}}_{s=0}^{\infty} (-\kappa_0 s + \epsilon\omega^t(s) | \partial_x \omega^t(s)), \\ \partial_t \partial_x \mathop{\mathfrak{t}}_{s=0}^{\infty} (\lambda^t(s)) &= \epsilon \mathop{\delta\mathfrak{t}}_{s=0}^{\infty} (-\kappa_0 s + \epsilon\omega^t(s) | \partial_t \partial_x \omega^t(s)) \\ &\quad + \epsilon^2 \delta^2 \mathop{\mathfrak{t}}_{s=0}^{\infty} (-\kappa_0 s + \epsilon\omega^t(s) | \partial_x \omega^t(s), \partial_t \omega^t(s)). \end{aligned}\right\} \tag{5.6}$$

Hence,

$$\partial_t \partial_x \mathop{\mathfrak{t}}_{s=0}^{\infty} (\lambda^t(s)) = \epsilon \mathop{\delta\mathfrak{t}}_{s=0}^{\infty} (-\kappa_0 s | \partial_t \partial_x \omega^t(s)) + O(\epsilon^2). \tag{5.7}$$

Equation (4.7) and the last equation of (5.5), when combined with (4.8), (4.9) and $(4.10)_1$, tell us that

$$\begin{aligned} \mathop{\delta\mathfrak{t}}_{s=0}^{\infty} (-\kappa s | \partial_t \partial_x^2 \omega^t(s)) &= \int_0^\infty G'(\kappa_0, s)\, \partial_t \partial_x^2 \omega^t(s)\, ds = \int_0^\infty G'(\kappa_0, s)\, [\partial_x^2 v_1(x, t-s) \\ &\qquad - \partial_x^2 v_1(x,t)]\, ds \\ &= \int_0^\infty G'(\kappa_0, s)\, \partial_x^2 v_1(x, t-s)\, ds + E(\kappa_0)\, \partial_x^2 v_1(x,t). \end{aligned} \tag{5.8}$$

If we now substitute (5.7) and (5.8) into (5.3) and neglect the term $O(\epsilon^2)$ in comparison with the terms linear in ϵ, we obtain the following equation for v_1:

$$E(\kappa_0)\,\partial_x^2 v_1(x,t) + \int_0^\infty G'(\kappa_0,s)\,\partial_x^2 v_1(x,t-s)\,ds = \rho\,\partial_t^2 v_1(x,t). \tag{5.9}$$

Integral-differential equations with this general form occur often in the linear theory of viscoelasticity. Of course, $E(\kappa_0)$ is the initial value $(4.10)_1$ of the stress-relaxation function (4.8) with $\kappa = \kappa_0$.

We emphasize that our derivation of (5.9) does not in any way assume that the underlying rate of steady shear, κ_0, must be small, although it does entail neglect of a term $O(\epsilon^2)$.

In addition to $(4.10)_2$ and $(4.11)_2$, let us here assume, as also seems natural, that $-G'(\kappa_0, s)$ is a non-negative, continuously differentiable, monotone-decreasing function of s on $(0, \infty)$. Then (5.9) is a stable, well-behaved equation which can be solved using methods familiar in linear viscoelasticity. For example, (5.9) has solutions of the form

$$v_1(x,t) = e^{-\zeta x}\cos(\omega(t - x/c)) = \mathscr{R}\{e^{-(\zeta+i\omega/c)x}\,e^{i\omega t}\}, \quad t\in(-\infty,\infty),\ x\in[0,\infty), \tag{5.10}$$

with ω, ζ and c real, positive, numbers. If v_1 in (5.2) obeys (5.10) then we say that the motion (3.1) describes an infinitesimal, spatially damped, sinusoidal shear wave (with frequency ω, attentuation ζ, speed c, and amplitude ϵ), superimposed on a simple shearing flow with velocity gradient κ_0. Such a periodic wave is called 'infinitesimal', because for a general incompressible fluid it can satisfy the dynamical equation (5.3) only if the term $O(\epsilon^2)$ in (5.7) is neglected. Substitution of (5.10) into (5.9) yields

$$-\rho\omega^2 = \left(\zeta + \frac{i\omega}{c}\right)^2 \{E(\kappa_0) + \bar{G}'(\kappa_0,\omega)\}, \tag{5.11}$$

where

$$\bar{G}'(\kappa_0,\omega) = \int_0^\infty G'(\kappa_0,s)\,e^{-i\omega s}\,ds. \tag{5.12}$$

For each frequency ω there is one pair of positive values of c and α satisfying (5.11) (see, for example, Berry 1958; Hunter 1960, §3; or Coleman & Gurtin 1965, §7):

$$\left.\begin{aligned} c = c(\kappa_0,\omega) &= \left(\frac{1}{\rho}\left|E(\kappa_0) + G'(\kappa_0,\omega)\right|\right)^{\frac{1}{2}} \sec\frac{\theta(\omega)}{2},\\ \zeta = \zeta(\kappa_0,\omega) &= \frac{\omega}{c(\kappa_0,\omega)}\tan\frac{\theta(\kappa_0,\omega)}{2},\\ \tan\theta(\kappa_0,\omega) &= \frac{\mathscr{I}\{E(\kappa_0)+\bar{G}'(\kappa_0,\omega)\}}{\mathscr{R}\{E(\kappa_0)+\bar{G}'(\kappa_0,\omega)\}} \quad (0\leqslant\theta\leqslant\tfrac{1}{2}\pi). \end{aligned}\right\} \tag{5.13}$$

We have observed that $G(\kappa_0, s)$ is, for each s, an even function of κ_0. Hence, (5.13) implies that, for each value of ω, the wave speed c and the attenuation ζ are even functions of κ_0. The equation (5.13) and the assumptions made about $G'(\kappa_0, .)$ imply that the high-frequency limits

$$c(\kappa_0,\infty) = \lim_{\omega\to\infty} c(\kappa_0,\omega), \quad \zeta(\kappa_0,\infty) = \lim_{\omega\to\infty}\zeta(\kappa_0,\omega), \tag{5.14}$$

exist and are given by

$$c(\kappa_0,\infty) = (E(\kappa_0)/\rho)^{\frac{1}{2}}, \quad \zeta(\kappa_0,\infty) = \frac{-G'(\kappa_0,0)}{2E(\kappa_0)\,c(\kappa_0,\infty)}. \tag{5.15}$$

Of course, these limiting values are also even functions of κ_0.

6. General theory of rectilinear shear-acceleration waves

Leaving behind the linearized expressions of the previous section, let us return to the exact theory of the dynamical equations (3.14). We are interested in the motion and stability of singular surfaces across which derivatives of the function v of (3.1) have jump discontinuities. Such singular surfaces may be called *rectilinear shear waves.*

Therefore, a rectilinear shear wave is, at each time t, a planar surface perpendicular to the x-axis of the fixed co-ordinate system in which (3.1) holds. If we write x_t for the value of the x co-ordinate on this surface, then the velocity of the wave is

$$u = u(t) = \frac{d}{dt}x_t. \tag{6.1}$$

We assume that $v(x,t)$ and all its derivatives $\partial_t^l\partial_x^k v(x,t)$ are continuous functions of the pair (x,t) whenever $(x,t) \neq (x_t,t)$ and that these quantities suffer, at worst, jump discontinuities $[v]$, $[\partial_t^l\partial_x^k v]$, across the wave.

Those rectilinear shear waves for which

$$[v] = 0, \quad \text{while} \quad [\partial_t v] \neq 0 \quad \text{and} \quad [\partial_x v] \neq 0, \tag{6.2}$$

we call *shear-acceleration waves.* Here we derive exact formulae for the velocity and the amplitude,

$$a = a(t) = [\partial_t v], \tag{6.3}$$

of shear-acceleration waves, assuming that the specific driving force α is constant in time and that the wave is moving into a region undergoing a steady rectilinear shearing flow. Thus, taking the wave velocity $u(t)$ to be positive, we suppose that for all $x \geqslant x_t$ and $t \geqslant 0$

$$v^x(x,\sigma) = 0, \quad v^y(x,\sigma) = v(x), \quad v^z(x,\sigma) = 0, \quad \text{for} \quad -\infty < \sigma \leqslant t. \tag{6.4}$$

Let $\partial_x v^t$ be the function on $(0,\infty)$ given by

$$\partial_x v^t(s) = \partial_x v(x,t-s) \quad (0 < s < \infty). \tag{6.5}$$

Ahead of the wave, by (6.4),

$$\partial_x v^t(s) = \kappa \quad (0 < s < \infty) \quad \text{with} \quad \kappa = \kappa(x) = \partial_x v(x), \tag{6.6}$$

and the function λ^t of (3.4) is given by (3.15); hence λ^t and $\partial_x v^t$ both lie in $\mathscr{H}$. Furthermore, an argument we have given for a related problem (Coleman & Gurtin 1965, pp. 253, 254) can be applied to show that the $\mathscr{H}$ valued-functions

$$(x,t) \to \lambda^t \quad \text{and} \quad (x,t) \to \partial_x v^t \tag{6.7}$$

are both continous across the acceleration wave with respect to the norm $\|.\|$ on $\mathscr{H}$. At the wave λ^t and $\partial_x v^t$ are given by

$$\lambda^t(s) = -s\kappa_t, \quad \partial_x v^t(s) = \kappa_t \quad (0 < s < \infty); \tag{6.8}$$

here κ_t denotes the value of $\partial_x v(x,t)$ just ahead of the wave:

$$\kappa_t = \kappa(x_t) = \lim_{x \to x_t^+} \frac{d}{dx} v(x), \tag{6.9}$$

with v the function shown in (6.4).

Employing (3.4) it is not difficult to show that the $\mathscr{H}$-valued function $t \to \lambda^t$ is differentiable when $x \neq x_t$; in fact,

$$\partial_t \lambda^t(s) = \partial_x v(x, t-s) - \partial_x v(x,t) = \partial_x v^t(s) - \partial_x v(x,t). \tag{6.10}$$

Hence the chain rule

$$\partial_t T^{xy}(x,t) = \partial_t \underset{s=0}{\overset{\infty}{\mathfrak{t}}} (\lambda^t) = \underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}} (\lambda^t(s) | \partial_t \lambda^t(s)) \tag{6.11}$$

and the linearity of $\delta\mathfrak{t}(.|.)$ in its second argument yield

$$\partial_t T^{xy}(x,t) = \underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}} (\lambda^t(s) | \partial_x v^t(s)) - \underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}} (\lambda^t(s)|1) \, \partial_x v(x,t). \tag{6.12}$$

Since $\delta\mathfrak{t}(.|.)$ is continuous over $\mathscr{H} \times \mathscr{H}$, and the $\mathscr{H}$-valued functions (6.7) are continuous across the wave, (6.12) and $(6.8)_1$ tell us that

$$[\partial_t T^{xy}] = E(\kappa_t)[\partial_x v], \tag{6.13}$$

where

$$E(\kappa_t) = - \underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}} (-\kappa_t s | 1). \tag{6.14}$$

In view of (4.13), we may call $E(\kappa_t)$ the *instantaneous tangent modulus at the wave.*

Since the smoothness assumption for $\mathfrak{t}$ laid down in §4 implies that the functional $\mathfrak{t}$ is continuous over $\mathscr{H}$, the equation $(3.5)_1$ and the continuity of the $\mathscr{H}$-valued function $(6.7)_1$ imply that T^{xy} is continuous across the wave. By Maxwell's theorem and the continuity of v and T^{xy}, we have

$$[\partial_t v] = -u[\partial_x v] \quad \text{and} \quad [\partial_t T^{xy}] = -u[\partial_x T^{xy}]. \tag{6.15}$$

Furthermore, since α is constant, it follows from the dynamical equation (3.10) that

$$[\partial_x T^{xy}] = \rho[\partial_t v]. \tag{6.16}$$

Equations (6.13), (6.15) and (6.16) imply that

$$(\rho - u^2 E_t)[\partial_t v] = 0. \tag{6.17}$$

Thus, *for the velocity u of a shear-acceleration wave entering a region where* (6.4) *holds, we have the formula*

$$u(t) = (E(\kappa_t)/\rho)^{\frac{1}{2}}, \tag{6.18}$$

with $E(\kappa_t)$ given by (6.14).

If we differentiate (6.12) with respect to t, employ (6.10), and recall that $\delta^2\mathfrak{t}(\lambda^t|.,.)$ is a symmetric bilinear form, we obtain

$$\begin{aligned}\partial_t^2 T^{xy}(x,t) = \Phi - \underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}} (\lambda^t(s)|1) \, \partial_t \partial_x v(x,t) + \underset{s=0}{\overset{\infty}{\delta^2\mathfrak{t}}} (\lambda^t(s) | \partial_x v^t(s), \partial_x v^t(s)) \\ + \underset{s=0}{\overset{\infty}{\delta\mathfrak{t}}} (\lambda^t(s)|1,1)(\partial_x v(x,t))^2 - 2 \underset{s=0}{\overset{\infty}{\delta^2\mathfrak{t}}} (\lambda^t(s)|1, \partial_x v^t) \, \partial_x v(x,t),\end{aligned} \tag{6.19}$$

with
$$\Phi = \frac{d}{d\tau} \underset{s=0}{\overset{\infty}{\mathfrak{t}}} (\lambda^t(s)|\partial_x v^\tau(s))|_{\tau=t}. \tag{6.20}$$

An argument we have given elsewhere (Coleman & Gurtin 1965, §2, equations (2.26)–(2.34)) shows that
$$[\Phi] = G'(\kappa_t, 0)[\partial_x v], \tag{6.21}$$

with $G'(\kappa_t, 0)$ equal to $G'(\kappa, 0)$ of (4.11) evaluated at the wave, i.e. at the value (6.9) of κ found just ahead of the wave. Therefore, (6.19), (4.17), (6.8), and the continuity of the functions (6.7) yield

$$[\partial_t^2 T^{xy}] = G'(\kappa_t, 0)[\partial_x v] + E(\kappa_t)[\partial_t \partial_x v] + \tilde{E}(\kappa_t)[(\partial_x v)^2] - 2\tilde{E}(\kappa_t)\kappa_t[\partial_x v], \tag{6.22}$$

and since†
$$[(\partial_x v)^2] = 2\kappa_t[\partial_x v] + [\partial_x v]^2, \tag{6.23}$$

we have
$$[\partial_t^2 T^{xy}] = G'(\kappa_t, 0)[\partial_x v] + E(\kappa_t)[\partial_t \partial_x v] + \tilde{E}(\kappa_t)[\partial_x v]^2, \tag{6.24}$$

with $\tilde{E}(\kappa_t)$ *the second-order instantaneous modulus evaluated at the wave*, i.e.

$$\tilde{E}(\kappa_t) = \underset{s=0}{\overset{\infty}{\delta^2 \mathfrak{t}}} (-\kappa_t s|1, 1). \tag{6.25}$$

A general formula (see, for example, Truesdell & Toupin 1960, equation (191.8); or Coleman & Gurtin, 1965, equation (2.4)) for the rate of change of jump discontinuities tells us that

$$\frac{d}{dt}[\partial_t T^{xy}] = [\partial_t^2 T^{xy}] + u[\partial_x \partial_t T], \tag{6.26}$$

and, by (6.3),
$$\frac{d}{dt} a(t) = [\partial_t^2 v] + u[\partial_x \partial_t v]. \tag{6.27}$$

Of course, (3.10) here yields
$$[\partial_t \partial_x T^{xy}] = \rho[\partial_t^2 v]. \tag{6.28}$$

By combining (6.26)–(6.28) with (6.15) and (6.16), we obtain the equation

$$-\frac{d}{dt}(\rho u a) = [\partial_t^2 T^{xy}] + \left(\frac{da}{dt} - u[\partial_t \partial_x v]\right) \rho u, \tag{6.29}$$

which, by (6.18), can be written

$$2\rho\sqrt{u}\frac{d}{dt}(a\sqrt{u}) = E(\kappa_t)[\partial_t \partial_x v] - [\partial_t^2 T^{xy}], \tag{6.30}$$

since $u = u(t)$ is assumed positive. Therefore, it follows from (6.24) and $(6.15)_1$ that

$$2\rho\sqrt{u}\frac{d}{dt}(a\sqrt{u}) = -G'(\kappa_t, 0)[\partial_x v] - \tilde{E}(\kappa_t)[\partial_x v]^2 \tag{6.31}$$

$$= G'(\kappa_t, 0)\frac{a}{u} - \tilde{E}(\kappa_t)\left(\frac{a}{u}\right)^2. \tag{6.32}$$

This last expression may be written as a differential equation of Bernoulli type:

$$\frac{db}{dt} + \gamma(t) b + \nu(t) b^2 = 0, \quad b = b(t) = a(t)\, u(t)^{\frac{1}{2}}, \tag{6.33}$$

† See, for example, Coleman, Greenberg & Gurtin (1966, equation (1.10*b*) with $f = g = \partial_x v$ and $f(t)^+ = g(t)^+ = \kappa_t$).

with
$$\gamma(t) = -\frac{G'(\kappa_t, 0)}{2E(\kappa_t)}, \quad \nu(t) = \frac{\tilde{E}(\kappa_t)}{2E(\kappa_t)\,u(t)^{\frac{3}{2}}}. \tag{6.34}$$

The differential equation (6.33) is easily solved to complete our proof of the following theorem. *Consider a shear-acceleration wave, of positive velocity $u(t)$, which since time $t = 0$ has been advancing into a region undergoing a steady rectilinear shearing flow* (6.4); *the amplitude $a(t)$ of such a wave obeys the formula*†

$$a(t)\,u(t)^{\frac{1}{2}} = \frac{a(0)\,u(0)^{\frac{1}{2}}\,e^{-\psi(t)}}{1 + a(0)\,u(0)^{\frac{1}{2}} \int_0^t \nu(\tau)\,e^{-\psi(\tau)}\,d\tau}, \quad \psi(t) = \int_0^t \gamma(\tau)\,d\tau, \tag{6.35}$$

with ν and γ given by (6.34).

If the fluid ahead of the wave is undergoing a rectilinear shearing flow that is not steady, i.e. if v in (6.4) depends on time, then an expression of the general form (6.35) still holds, but $\gamma(t)$ and $\nu(t)$ are not given by (6.34). (Cf. Coleman, Greenberg & Gurtin 1966, §3).

7. Waves entering regions undergoing simple shearing flow

We here examine the special case in which the equations (6.4), which give the velocity field ahead of the wave, describe a simple shearing flow and therefore reduce to

$$v^x(x,\sigma) = 0, \quad v^y(x,\sigma) = V + \kappa_0 x, \quad v^z(x,\sigma) = 0, \quad \text{for} \quad -\infty < \sigma \leqslant t, \quad x \geqslant x_t, \tag{7.1}$$

with V and κ_0 constants independent of x and t. In this case, the expression (6.18) for the velocity u of a shear-acceleration wave reduces to

$$u = (E(\kappa_0)/\rho)^{\frac{1}{2}}, \tag{7.2}$$

and thus u is constant in time.

If we suppose that the shear-acceleration wave has been advancing into the region (7.1) since time $t = 0$, then its amplitude $a(t)$ obeys, by (6.35) and (7.2), the following simplified form of (6.35):

$$a(t) = \frac{\lambda}{\left(\frac{\lambda}{a(0)} - 1\right)e^{\gamma t} + 1}; \tag{7.3}$$

here

$$\gamma = \gamma(\kappa_0) = \frac{-G'(\kappa_0, 0)}{2E(\kappa_0)} = \text{const.}, \quad \lambda = \lambda(\kappa_0) = \frac{uG'(\kappa_0, 0)}{\tilde{E}(\kappa_0)} = \text{const.}; \tag{7.4}$$

$E(\kappa_0)$ is the initial value $(4.10)_1$, and $G'(\kappa_0, 0)$ is the initial slope $(4.11)_1$ of the stress-relaxation function for shearing perturbations about the steady shear κ_0, while $\tilde{E}(\kappa_0)$ is the second-order instantaneous modulus (4.17) for shearing perturbations about κ_0.

If the fluid be such that $G'(\kappa_0, 0) = 0$, the equation (7.3) becomes‡

$$a(t) = \frac{a(0)}{1 + \nu a(0)\,t}, \quad \nu = \frac{\tilde{E}(\kappa_0)}{2u E(\kappa_0)}. \tag{7.5}$$

† The derivation we have given here appears to us more direct and transparent than the proofs given by Coleman, Greenberg & Gurtin (1966) for their theorems 3.1, 7.1 and 7.2, which, taken together, also imply that (6.35) holds for fluids obeying the postulate of fading memory.

‡ Of course, for a perfect fluid not only is $G'(\kappa_0, 0)$ zero but so also is $E(\kappa_0)$, and (7.5) reduces to $a(t) \equiv a(0)$ while (6.18) becomes $u \equiv 0$.

For the rest of this discussion we shall assume that the inequality in $(4.11)_2$ is strict:

$$G'(\kappa_0, 0) < 0. \tag{7.6}$$

The equations (5.15), (7.2) and (7.4) yield the following relations between the parameters describing shear-acceleration waves and those characterizing infinitesimal sinusoidal waves (5.10) superimposed on the same underlying simple shearing flow:

$$u(\kappa_0)^2 = \lim_{\omega\to\infty} c(\kappa_0, \omega)^2, \quad \gamma(\kappa_0) = \lim_{\omega\to\infty} \zeta(\kappa_0, \omega)\, c(\kappa_0, \omega). \tag{7.7}$$

We note that the theory of infinitesimal sinusoidal waves contains no analogue of the parameter $1/\lambda$ occurring in the *exact* formula (7.3). This is not surprising, for $1/\lambda$ is proportional to

$$\tilde{E}(\kappa_0) = \overset{\infty}{\underset{s=0}{\delta^2 \mathfrak{t}}} (-s\kappa_0 | 1, 1), \tag{7.8}$$

and such second derivatives of $\mathfrak{t}$ do not occur in the first-order term exhibited in (5.7).

When $\tilde{E}(\kappa_0) = 0$, (7.3) reduces to

$$a(t) = a(0)\, e^{-\gamma t}. \tag{7.9}$$

Of course, it follows from (7.4), (7.6) and $(4.10)_2$ that $\gamma = \gamma(\kappa_0)$ is positive.

As we saw in (4.19), $\tilde{E} = 0$ when $\kappa_0 = 0$. Thus, the amplitude of a shear-acceleration wave propagating into a region at equilibrium decays to zero exponentially. (Cf. Coleman & Gurtin 1965, remark 6.1.)

Since $\tilde{E}$ may be non-zero when $\kappa_0 \neq 0$, if the region ahead of the wave is not in a state of equilibrium, the wave-amplitude need not decay to zero. In fact, when $\tilde{E}(\kappa_0) \neq 0$, the number $|\lambda|$ plays the role of a *critical amplitude*. Because we take u to be positive and assume $(4.10)_2$ and (7.6), the equation (7.3) has the following properties. If $|a(0)| < |\lambda|$ or if $\operatorname{sgn} a(0) = \operatorname{sgn} \tilde{E}$, then $a(t) \to 0$ monotonically as $t \to \infty$. If $a(0) = \lambda$, then $a(t) \equiv a(0)$. On the other hand, if both $|a(0)| > |\lambda|$ and $\operatorname{sgn} a(0) = -\operatorname{sgn} \tilde{E}(\kappa_0)$, then $|a(t)| \to \infty$ monotonically and in a *finite* time t_∞, given by

$$t_\infty = -\frac{1}{\gamma} \ln \left(1 - \frac{\lambda}{a(0)}\right). \tag{7.10}$$

Let us now further assume that

$$\kappa \neq 0 \Rightarrow \tilde{E}(\kappa) \neq 0, \tag{7.11}$$

so that either (4.20) or (4.21) holds. Then, by (7.4), (7.2), (7.6) and (4.10),

$$\kappa_0 \neq 0 \Rightarrow 0 < |\lambda(\kappa_0)| < \infty; \tag{7.12}$$

i.e. when the simple shearing flow (7.1) does not reduce to a state of rest, the critical amplitude $|\lambda(\kappa_0)|$ is finite and non-zero. Thus, by the observations made above, when $\kappa_0 \neq 0$, if the magnitude of the wave-amplitude $a(0)$ is sufficiently small or if $a(0)$ has the same sign as $\tilde{E}(\kappa_0)$, an acceleration wave entering a region undergoing the motion (7.1) is 'damped out'. If, however, $a(0)$ exceeds $\lambda(\kappa_0)$ in magnitude and has its sign opposite to that of $\tilde{E}(\kappa_0)$, then the wave 'blows up', i.e. achieves an infinite amplitude in a finite time t_∞; we suppose, albeit a proof is lacking, that the approach of $a = [\partial_t v]$ to infinity signifies the appearance of a discontinuity in v at time t_∞. A surface across which the function v in (3.1) suffers a jump discontinuity $[v] \neq 0$ is called a *vortex sheet.*

If a vortex sheet is formed at $t = t_\infty$ from a shear-acceleration wave, then we expect $[v]$ for $t > t_\infty$ to agree in sign with $[\partial_t v]$ for $t < t_\infty$, and we can easily show that the sign of $[\partial_t v]$ is *determined*, as shown in table 1, by two pieces of information: (a) knowledge of whether the fluid is shear-stiffening ($\kappa_0 \tilde{E}(\kappa_0) > 0$) or shear-softening ($\kappa_0 \tilde{E}(\kappa_0) < 0$), and ($b$) knowledge of whether the wave is propagating in the direction of increasing shear-velocity ($\kappa_0 > 0$) or decreasing shear-velocity ($\kappa_0 < 0$). This is a consequence of the fact that we can have $a(t) \to \infty$ only if $\operatorname{sgn} a(0) = -\operatorname{sgn} \tilde{E}(\kappa_0)$. If we suppose the fluid to be shear-stiffening and let κ_0 be positive, then $\tilde{E}(\kappa_0)$ is positive and $a(t) \to \infty$ only if $a(0)$ is negative, in agreement with the first line of table 1. Similarly, if we again assume the fluid to be shear-stiffening, but let κ_0 be negative, then we have $\tilde{E}(\kappa_0)$ negative and $a(t) \to \infty$ only if $a(0)$ is positive, in agreement with the second line of table 1. Analogous arguments for shear-softening fluids verify the third and fourth lines.

Type of fluid	Sign of κ_0	Required sign of $[\partial_t v]$ to have $\lvert[\partial_t v]\rvert \to \infty$
Shear-stiffening	+	−
Shear-stiffening	−	+
Shear-softening	+	+
Shear-softening	−	−

TABLE 1. Determination of the sign of the amplitude of those shear-acceleration waves which 'blow up'. We assume the co-ordinate system is so chosen that the wave velocity is positive.

By (7.4), (7.2) and (4.19) the critical amplitude $|\lambda(\kappa_0)|$ is infinite when $\kappa_0 = 0$, and, if we assume (7.11), $|\lambda(\kappa_0)|$ is finite for $\kappa_0 \neq 0$. This suggests that $|\lambda(\kappa_0)|$ should be a monotone decreasing function of κ_0 near $\kappa_0 = 0$, although a proof is lacking.

We may summarize as follows the observations made so far in this section. Consider a shear-acceleration wave propagating into a region undergoing the simple shearing motion (7.1). Albeit it is impossible for such a wave to grow in amplitude when $\kappa_0 = 0$, the wave can achieve infinite amplitude and form a vortex sheet if $\kappa_0 \neq 0$, provided $[\partial_t v]$ is of proper sign and exceeds in magnitude a critical amplitude. We expect the critical amplitude to decrease as the rate of shear κ_0 ahead of the wave increases, at least for κ_0 near to zero. Furthermore, table 1 tells us that for a shear-stiffening fluid a shear-acceleration wave moving in the direction of increasing (decreasing) shear velocity v can achieve infinite amplitude only if $[\partial_t v]$ is negative (positive). On the other hand, if the fluid is shear-softening, a wave moving in the direction of increasing (decreasing) velocity can achieve infinite amplitude only if $[\partial_t v]$ is positive (negative).

If the region ahead of the wave is undergoing a steady rectilinear shearing flow

(6.4) for which $\kappa = dv(x)/dx$ does not reduce to a constant κ_0 independent of x, the amplitude formula (6.25) does not reduce to (7.3). In such cases, an analysis of the stability of the wave becomes more difficult than here, but can be performed† if E, $\tilde{E}$, and $G'(0)$ are known ahead of the wave, i.e. if $E(\kappa)$, $\tilde{E}(\kappa)$ and $G'(\kappa, 0)$ are known as functions of κ and if κ is known as a function of x. Of course, the most important case of a rectilinear shearing flow with the driving force α constant in time but not zero is steady channel flow, for which κ is given by (3.23). If we assume, as is usual, that τ, and hence τ^{-1}, is a monotone increasing function, then it follows from (3.23) that $|\kappa|$ is a maximum at the bounding surfaces $x = \pm\frac{1}{2}d$. From this and the observations made above for simple shearing flow, we may conclude that a shear-acceleration wave propagating into a fluid undergoing steady channel flow is more likely to transform into a vortex sheet when it is near rather than far from the bounding surfaces.

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REFERENCES

Bagley, E. B. 1957 *J. Appl. Phys.* **28**, 624.

Bagley, E. B. 1963 *J. Appl. Polymer Sci.* **7**, 87.

Benbow, J. J., Charley, R. V. & Lamb, P. 1961 *Nature, Lond.* **192**, 223.

Berry, D. S. 1958 *Phil. Mag.* **25**, 100.

Coleman, B. D., Greenberg, J. M. & Gurtin, M. E. 1966 *Arch. Rational Mech. Anal.* **22**, 333.

Coleman, B. D. & Gurtin, M. E. 1965 *Arch. Rational Mech. Anal.* **19**, 239.

Coleman, B. D., Markovitz, H. & Noll, W. 1966 *Viscometric Flows of Non-Newtonian Fluids*. Berlin, Heidelberg, New York: Springer.

Coleman, B. D. & Noll, W. 1959 *Arch. Rational Mech. Anal.* **3**, 289.

Coleman, B. D. & Noll, W. 1960 *Arch. Rational Mech. Anal.* **6**, 355.

Coleman, B. D. & Noll, W. 1961*a* *Ann. N.Y. Acad. Sci.* **89**, 672.

Coleman, B. D. & Noll, W. 1961*b* *Rev. Mod. Phys.* **33**, 239, **36**, 1103.

Dunwoody, J. & Dunwoody, N. T. 1965 *Int. J. Eng. Sci.* **3**, 417.

Hunter, S. C. 1960 Viscoelastic Waves. In *Progress in Solid Mechanics*, vol. I, ed. by I. N. Sneddon and R. Hill. Amsterdam: North-Holland.

Nason, H. K. 1945 *J. Appl. Phys.* **16**, 338.

Noll, W. 1958 *Arch. Rational Mech. Anal.* **2**, 197.

Spencer, R. S. & Dillon, R. E. 1949 *J. Colloid Sci.* **4**, 241.

Tordella, J. P. 1956 *J. Appl. Phys.* **27**, 454.

Tordella, J. P. 1957 *Trans. Soc. Rheology*, **1**, 203.

Tordella, J. P. 1963 *J. Appl. Polymer Sci.* **7**, 215.

Truesdell, C. & Noll, W. 1965 The non-linear field theories of mechanics. In the *Encyclopedia of Physics*, vol. III/3. Ed. by S. Flügge. Berlin, Heidelberg, New York: Springer.

Truesdell, C. & Toupin, R. 1960 The classical field theories of mechanics. In the *Encyclopedia of Physics*, vol. III/1, pp. 226–902. Ed. by S. Flügge. Berlin, Göttingen, Heidelberg: Springer.

Varley, E. 1965 *Arch. Rational Mech. Anal.* **19**, 215.

† For a detailed discussion, see Coleman, Greenberg & Gurtin (1966, §3).